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17. ABSTRACT (MAX. 250 WORDS)

The program attempts to teach new mathematical concepts to elementary school children without the burden of learning a completely new vocabulary. The program will seek a high degree of pupil involvement brought about by fostering a spirit of inquiry and investigation. The absence of "answers" from an authority will be noted throughout the program. Five games are employed to develop the concepts of number lines, equations, functions, graphing, and signed numbers.

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1965 HARVARD-BOSTON SUMMER PROGRAM

MATHEMATICS CURRICULUM

The following sequence of topics is suggested primarily because it contains and previews important mathematical ideas. Concrete in the sense of making comprehensible some sophisticated and abstract concepts, it affords an opportunity for all to profit from an experience with the "new mathematics" without the burden of learning a completely new vocabulary.

While computational skills as such are not the focus of this program, it is believed that motivation for developing an adeptness with "number facts" is provided via a different and exciting encounter with mathematics. Furthermore, ample material is here for the imaginative though heavily scheduled teacher to use in his classroom throughout the school year. It should also be noted that no additional equipment is needed.

In keeping with the goals of the summer program, the emphasis is placed on a high degree of pupil involvement--brought about by fostering a spirit of inquiry and investigation, with a respectful encouragement of pupil responses. The absence of "answers" from authority will be noted throughout; pupils are encouraged to seek new methods of solution and to pose new questions.

The material is divided into five sections:

I. "Hops" on The Number Line
II. Grid Game
III. Extended Number Line
IV. Extended Grid Game
V. Open Sentences and Graphing

The material within each section has been developed sequentially with a strong degree of flexibility as to how far each idea is carried. The topics are intended to be open-ended. Advanced questions and enrichment materials are provided and indicated by an asterisk(*). At the end of each section, references are made to pertinent source material, with particular reference in Sections I and III to the University of Illinois Arithmetic Project under the direction of Professor David Page and in Sections II, IV, and V to the Madison Project under the direction of Professor Robert B. Davis.

The five sections are developmental and interrelated. It may be feasible for some pupils to cover only the material in the first four sections. Even without Section V, the pupils will feel an accomplishment of having completed an important sequence of ideas. Section V provides a concretization of these ideas which it is hoped that most students will have the opportunity of experiencing.

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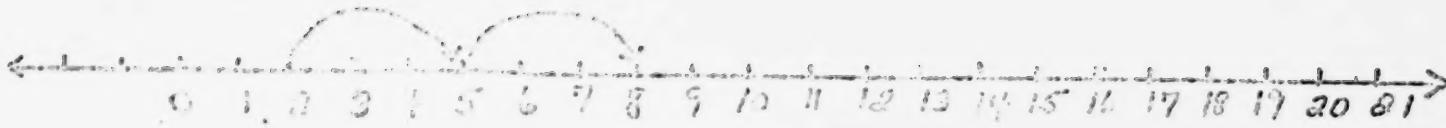
A. "Hops" on the number line

A.



Questions such as:

Where do you land next?	(11)
Starting at 11, where will you be after one more hop?	(14)
... after two more hops?	(17)
... after three more hops?	(20)
... after five more hops?	(26)
If you took two hops and landed at 18, where did you start?	(12)
If you start at 9, will you ever land at 19? Why or why not?	
*Suppose you start at 0 and I start at 1. Will we ever land at the same spot at the same time?	

B. Notation

Rule A: [2] [21] × 3
i.e. 2 "hops into" 5 by Rule A

Rule B: [9] [2] − 4
i.e. 9 "hops into" 5 by Rule B

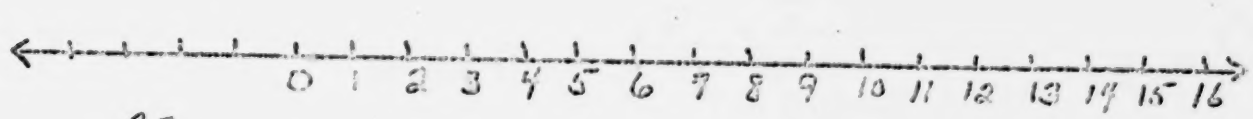
This suggests the following notation:

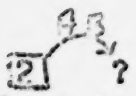
[9] [9] × 3
9 "hops into" 12 by Rule A

[9] [2] − 4
9 "hops into" 5 by Rule B

C. Combinations of rules

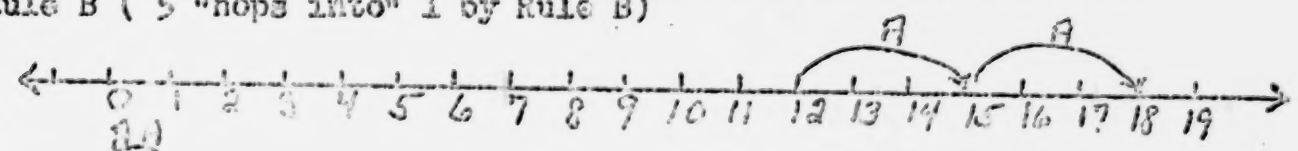
1.

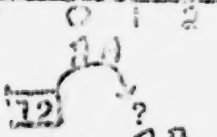




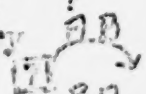
Start at 2, use Rule A (2 "hops into" 5 by Rule A), then use Rule B (5 "hops into" 1 by Rule B)

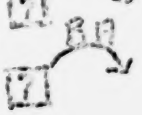
2.





3. Now try

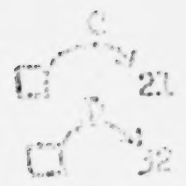




and

4. Will these results always be the same?
5. Can you give a single rule that is the same as AB?

D. [illegible] I [illegible] a problem via combinations of rules constructed in such a way that the pupil can find pitfalls and shortcuts out, including some for which obvious shortcuts don't hold.



(36)
(26)
(26)
(34)
(34)
(39)
(25)

Note: The last two are not the same. Encourage the pupils to find out why.

(15)
(23)
(40)
(34)
(40)

Note the last three examples.

E. At this point, pupils should make their own rules. Combinations of various pupils' rules could then be tried in order to see which ones allow shortcuts-- and why.

F. References.

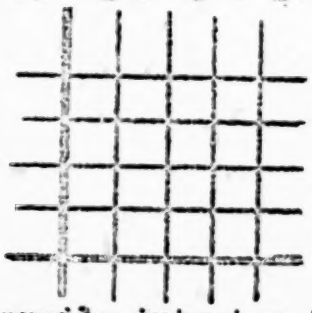
Hohn, Frank E., "Teaching Creativity in Mathematics", reprinted from "The Arithmetic Teacher", March, 1961.

Nagel, David, Number Line, Fractions and Fundamental Topics, Macmillan, 1964.

II. Grid Game

A. How to play the game

Put the following 5 by 5 grid on the board.



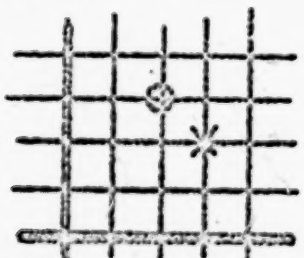
Divide the pupils into two teams (down the middle of the room, widening the aisle between the two teams if possible).

Name one team "x" and the other one "o".

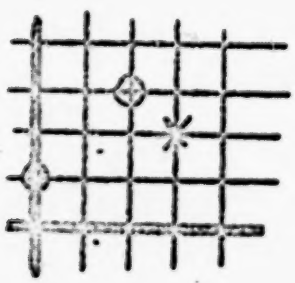
Ask someone from the "x" team to give two numbers. Suppose the numbers 7,10 are given.

Without saying anything further, ask someone from the "o" team to give two numbers. Suppose the numbers 2,3 are given. Place an o on the grid as follows:

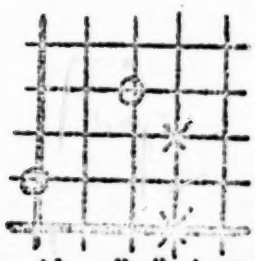
Then ask someone from the "x" team to give two more numbers. Suppose the numbers 3,2 are given. Place an x on the grid as follows:



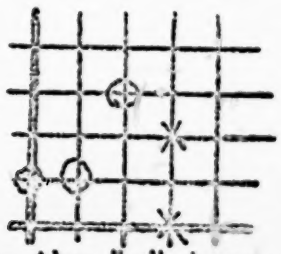
Suppose the "o" team then gives 0,1. Place an o on the grid as follows:



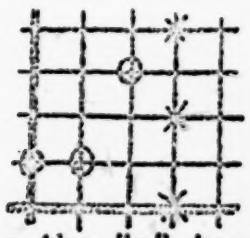
Suppose the "x" team gives 3,0.



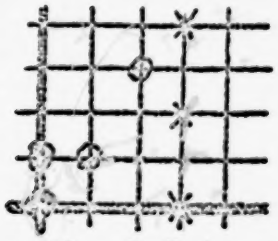
Suppose the "o" team gives 1,1.



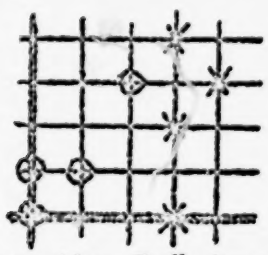
Suppose the "x" team gives 3,4.



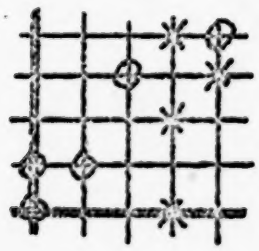
Suppose the "o" team gives 0,0.



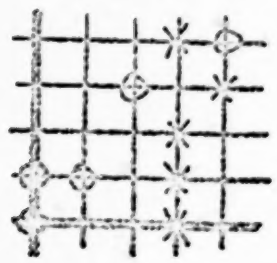
Suppose the "x" team gives 4,3.



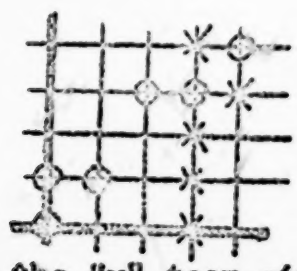
Suppose the "o" team gives 4,4.



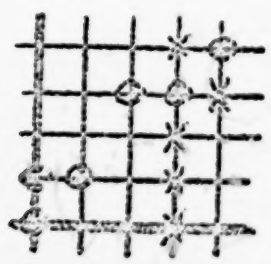
Suppose the "x" team gives 3,2.



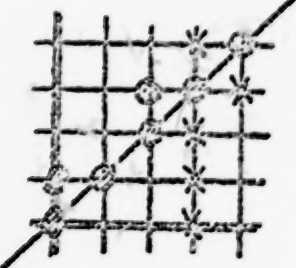
Suppose the "o" team gives 3,3. (It looks as if the "o" team has caught on to the game!)



Suppose the "x" team gives 1,1. (Pass to the "o" team.)



Suppose the "o" team gives 2,2.

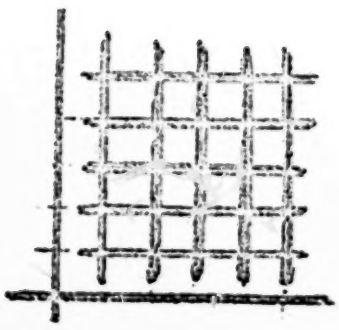


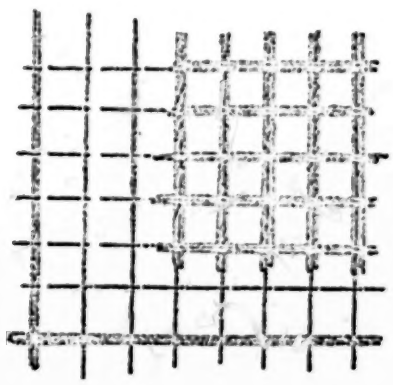
The "o" team won!!

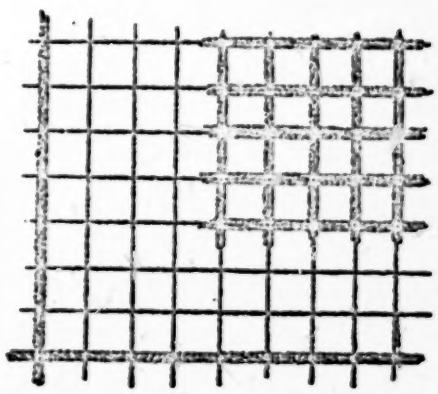
Repeat the game until each team is able to prevent the other team from winning. Also you may wish to break the class into smaller teams (even into pairs), and let the pupils mark their own moves.

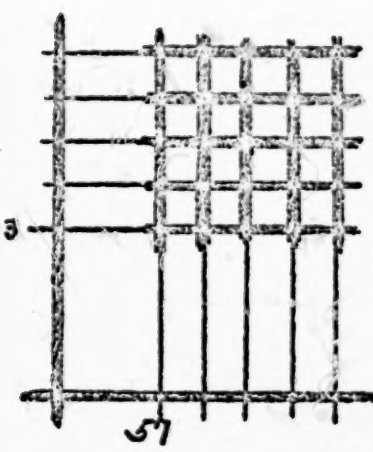
B. Modifications of the game

1. Place the grid in different regions to involve larger numbers, for example:





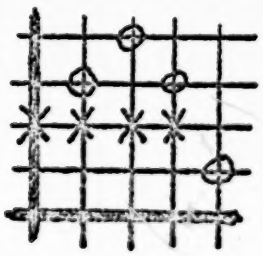




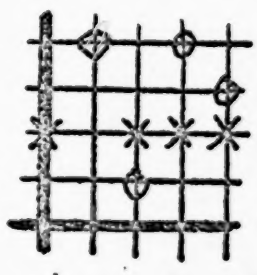
* Even get to

It should be noted that playing "5 in a line" is easy for defense, hard for offense. The game often leads to a tie. It is suggested that you move quickly into the new version below.

2. a. Still play on a 5 by 5 grid but change a "win" to 4 <u>consecutive</u> dots rather than "5 in a line".



A "win"



Not a "win"

b. Plan "4 in a line" in various regions to involve larger numbers.

* 2. Play "3 in a line" on a 5 by 5 grid in various regions.

C. It becomes obvious that playing "4 in a line" is much more exciting than playing "5 in a line". The question can be raised as to why this is so. It is not a difficult problem for the pupils to actually count the number of possible ways to win in each case.

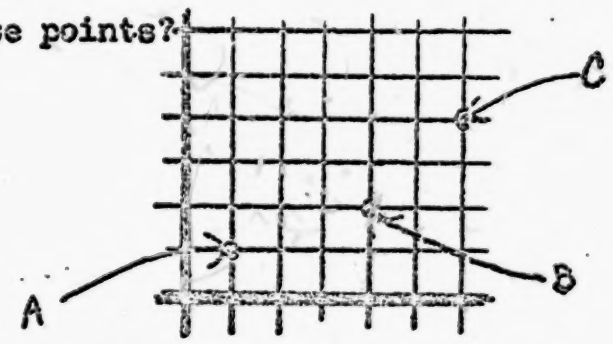
Questions such as:

How many possible wins are there for "5 in a line" on a 5by5 grid? (12)
How many possible wins are there for "4 in a line" on a 5by5 grid? (28)
* How many possible wins are there for "3 in a line" on a 5by5 grid?
* How many possible wins are there for "4 in a line" on a 4by4 grid?
* How many possible wins are there for "3 in a line" on a 4by4 grid?
* Which game is the most fun? Why?
* What would happen if you played "2 in a line"?

D. Ask questions such as:

1. What numbers describe these points?

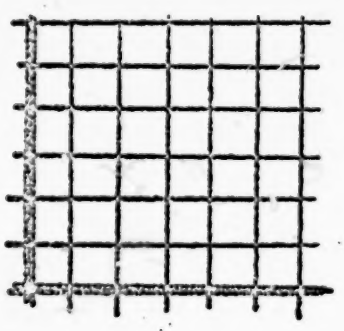
A _____?
B _____?
C _____?



A (1,1)
B (4,2)
C (6,4)

2. Can you indicate these pairs of numbers on the grid?

D (2,4)
E (4,6)
F (5,3)
G (0,0)



* E. Optional-- Point Set Game

See "Discovery in Mathematics", Robert B. Davis, Chapter 7, pp. 55-66.

F. References.

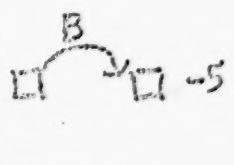
Davis, Robert B., "The Evolution of School Mathematics", reprinted from "Journal of Research in Science Teaching", 1963.

" Madison Project: Supplementary Modern Mathematics, In-Service Course for Teachers", PP. I-12--I-17, II-12--II-18.

III. Extended Number Line

Where do I land if I start at 10? (7)
If I hop again? (5)
And again? (4)
And again? (2)
And again? (0)
What if I start at 9 and hop the same number of times, - where will I land? Answers will vary, from "can't" to "one below zero" to "right here"(pointing).
(To last pupil) What if I now use Rule A again?

Note: it is probably wise to avoid terminology for negative numbers until pupils become adept at simple problems involving them. Care should be taken to distinguish "1", "negative one", from the operation symbol "-", "minus", by placing the negative sign high beside the number.

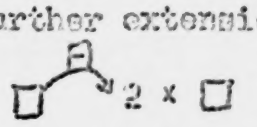
B. Introduce another rule, such as 

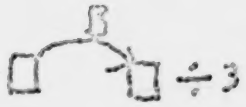
Starting at 7, use A, then B. (0)
Starting at 19, use B, then A, then A. (10)
Starting at 30, use B, then A, then A. (21)
Can you write a single rule for this?
*Starting at ?, use B. (-6)

C. Inverses

Write a rule which "undoes" A, call it C.
Use A, then C, for several numbers to see if your C always undoes A.
Write a rule which "undoes" B, call it D.
Try B, then D, for several numbers.
How about using D, then B, does B "undo" D?

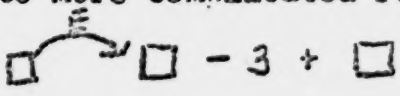
D. Further extension



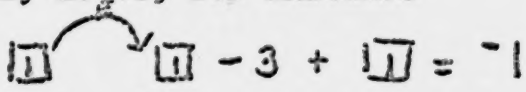


1. Starting at 5, use A. (10)
   Starting at 9, use B. (3)
   Starting at 6, use A, then B. (4)
   *Starting at 0, use B. (0)
2. Write a rule which undoes A, call it C.
   Write a rule which undoes B, call it D.
3. Starting at 6, use C. (3)
   Starting at 8, use C. (4)
   Starting at 7, use C. If you can't find the point, can you guess about where it is?

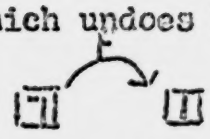
*E. Extend to more complicated rules, such as

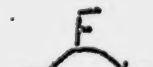


Encourage the use of several numbers before generalization can be accepted. For Rule E, above, for instance



A rule which undoes E in this case could be F such that



(It is possible to give many rules for F, such as 

It should then be suggested that 2 be tried.)

F. Enrichment material.

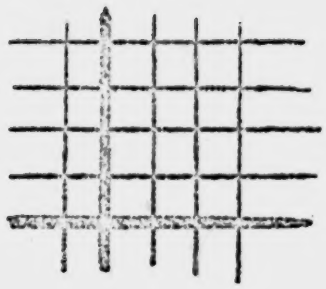
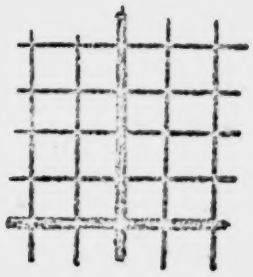
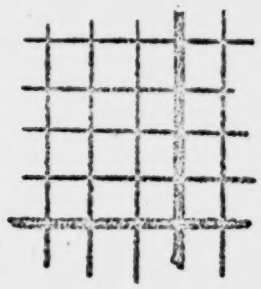
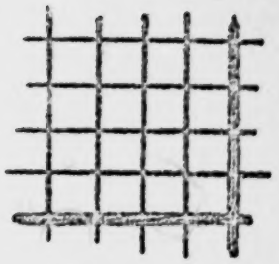
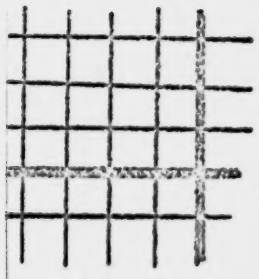
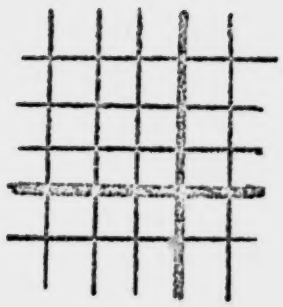
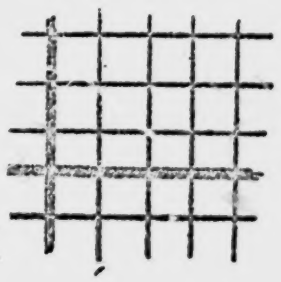
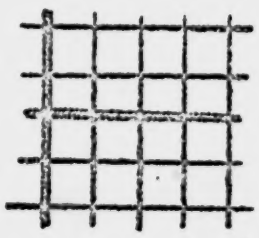
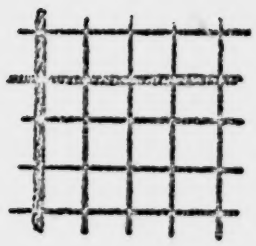
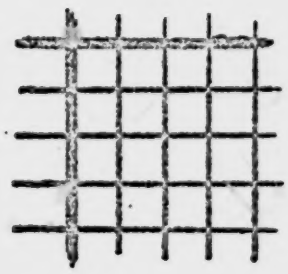
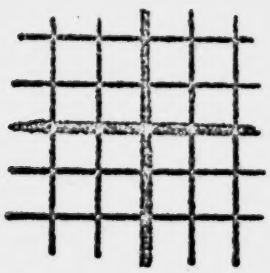
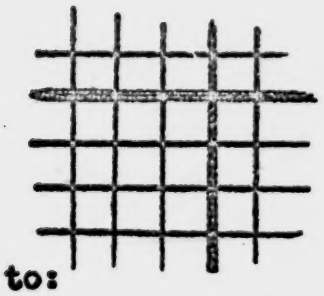
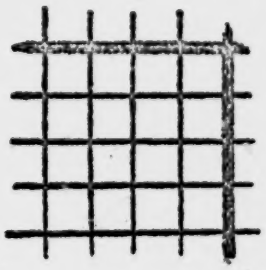
1. "Pebbles - in - the - bag" game. (model for signed numbers) Madison Project, "In-Service Course for Teachers", pp. I-4-I-10, I-20-I-22. Also film excerpt, "Pebbles in the Bag".

G. References.

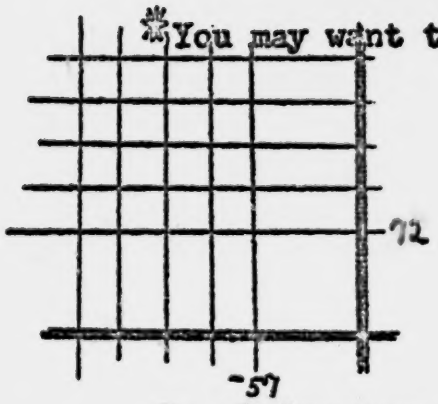
Page, David, Number Line, Fractions and Fundamental Topics, Macmillan, 1964.

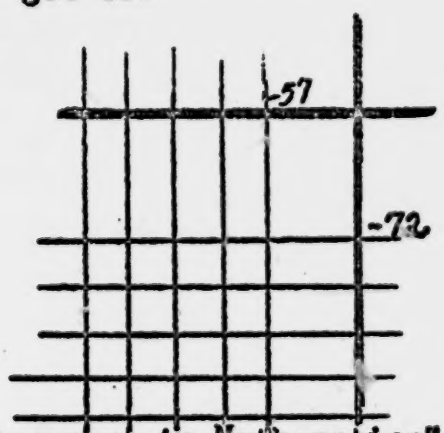
IV. Extended Grid Game

A. Play "5 in a line" (then "4 in a line")

*You may want to gradually get to:





B. References.

Davis, Robert B., "Discovery in Mathematics", Chapter 9, pp. 71-74.

V. Open Sentences and Graphing

A. Questions such as:

What number can we write in the □ to make a true statement?

2 + □ = 5	{3}	(No method of solution should be imposed. Pupils should be encouraged to find and test their own methods.)
3 + □ = 7	{4}	
8 + □ = 9	{1}	
5 + □ = 11	{6}	
□ + 6 = 26	{20}	
□ + 20 = 50	{30}	
□ + □ = 8	{4}	(Establish the ground rule that whatever number we put into the first " □ ", we will put this same number into all occurrences of the " □ " in the same sentence.)
□ + □ = 26	{13}	
□ + □ = 5	{2 1/2}	(The pupils may wish to approach this type of problem by finding a number which is too small, say 2, and a number which is too large, say 3, and then "close in" on the correct value. Some may want to use a number line.)
□ + □ + □ = 10	{3 1/3}	
15 - □ = 7	{8}	
□ - 5 = 8	{13}	
2 x □ = 6	{3}	
5 x □ = 30	{6}	
□ x 4 = 28	{7}	
3 x 5 = □	{15}	
3 + (2 x □) = 9	{3}	
5 + (2 x □) = 17	{6}	
3 + (2 x □) = 53	{25}	
3 + (2 x □) = 8	{2 1/2}	
□ + 5 = □ + □	{5}	

* □ + 7 = □ + 3 + 2 + □ {2.}

* □ + 1 + 3 = 2 + □ + 2 {All numbers}

* □ + 6 + 3 = 12 - 5 + □ {No number}

B. For problems which the pupils find difficult, keeping a record of the result of the numbers which have been tried will often help them obtain a solution. They should be encouraged to look for patterns.

Example:

□ + □ + 3 + □ = □ + 21

Try 1: 1 + 1 + 3 + 1 = 1 + 21

6 = 22

Try 4: 4 + 4 + 3 + 4 = 4 + 21

15 = 25

Try 6: 6 + 6 + 3 + 6 = 6 + 21

21 = 27

Try 10: 10 + 10 + 3 + 10 = 10 + 21

33 = 31

Try 9: 9 + 9 + 3 + 9 = 9 + 21

30 = 30 √

C. Open Sentences with two different frames.

Example: □ + △ = 7 (Establish the rule that if the frames are different shapes, then the numbers may be either the same or different. Keep the rule that if the shapes are the same then the numbers that you substitute must be the same.)

Play the following game: I will give you a number to substitute into the " □ "; you give me a number to substitute into the " △ " to make the statement true. Get:

□	△
0	7
1	6
2	5

3	4
5	2
6	1
3 1/2	3 1/2

Etc.

For the open sentence $3 + (1 \times \square) = \triangle$, get

$\square$	$\triangle$
0	3
1	4
2	5
3	6
4	7
5	8

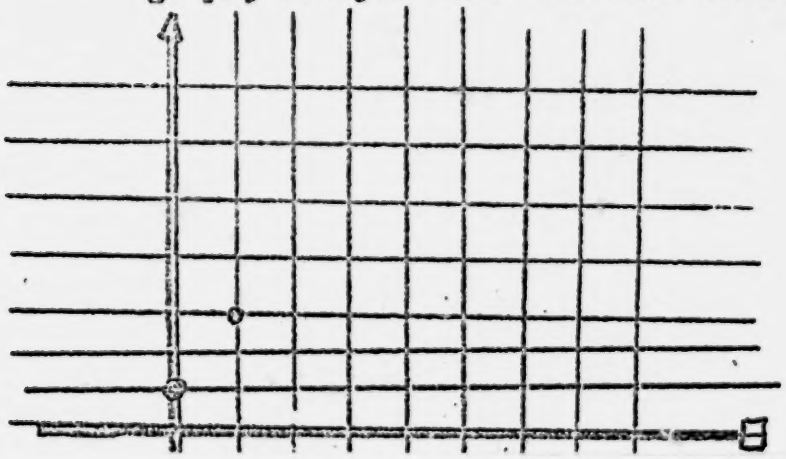
Etc.

For the open sentence $(2 \times \square) + 1 = \triangle$, get

$\square$	$\triangle$
0	1
1	3
2	5
3	7
4	9

Etc.

Then ask: The first two pairs of numbers have been marked on this graph; can you mark the next three pairs on the graph?



for the open sentence □ + △ = 2, get

□	△
0	2
1	1
2	0
3	-1
4	-2

Etc.

Ask questions such as:

Can you mark these points on a graph?
Have you discovered any secret patterns?
Looking at the graph, can you give another pair of numbers that make the statement true? Try them.

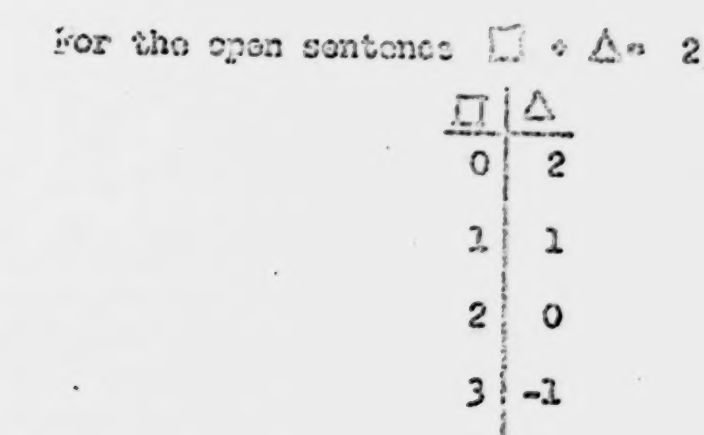
* How can you tell which way the line will slant?

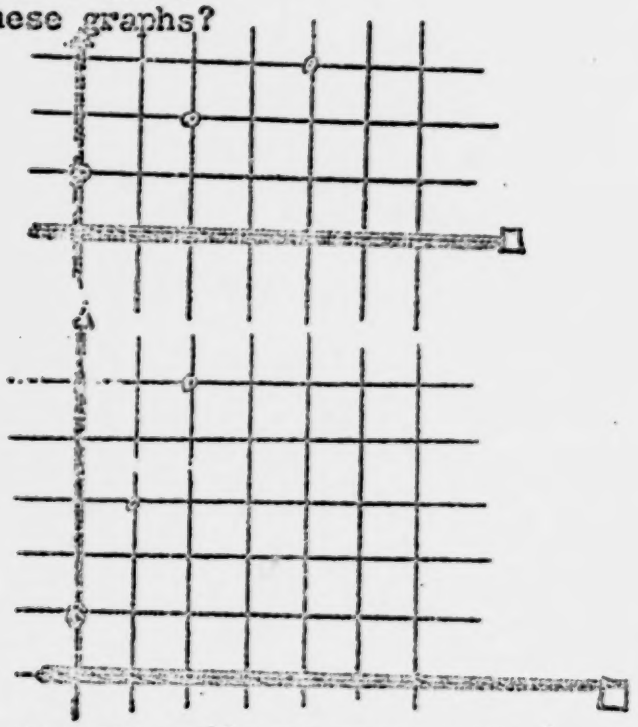
* Can you match these sentences and these graphs?

△ = (2 x □) + 1

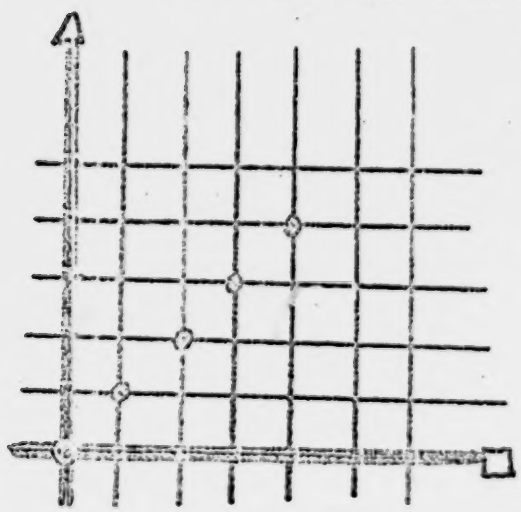
△ = □ + 1

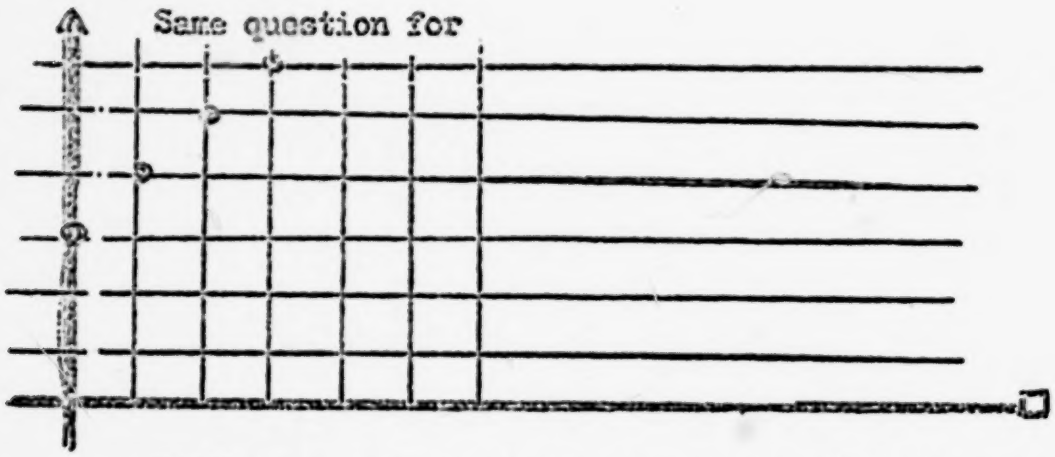
2 x △ = □ + 2





* Given a graph such as the following, can you write an open sentence which is made true by these pairs of numbers?





Will every open sentence give points that lie on a straight line?

※ D. Play the grid game and see if they can give a correct open sentence of the line which contains the "win."

E. References.

"Arithmetic with Frames", University of Illinois Arithmetic Project, reprinted from the Twenty-Seventh Yearbook, National Council of Teachers of Mathematics, Washington, D.C., 1963.

Davis, Robert B., "Math Takes a New Path," reprinted from "The PTA Magazine", February, 1963.

"A Brief Introduction to Materials and Activities"

"Madison Project: Supplementary Modern Mathematics, In-Service Course for Teachers," pp. II-2--II-11, II-20--II-27, III-2--III-12, III-13--III-17.

"Discovery in Mathematics", Chapter I, pp. 25-34
Chapter 3, pp. 38-42
Chapter 10, pp. 75-78
Chapter 11, pp. 79-94
Chapter 13, pp. 99-100
Chapter 15, pp. 103-107
Chapter 17, pp. 110-119
Chapter 18, pp. 120-124
Chapter 19, pp. 125-129
Chapter 30, pp. 191-198